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Supplement to
ON WAVEWISE ENTROPY INEQUALITIES FOR HIGH RESOLUTION SCHEMES I: THE SEMIDISCRETE CASE

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5. OVERVIEW OF THE PROOF OF THEOREM 3.13

The goal of this supplement is to prove the third WEI criterion, Theorem 3.13. We argue by contradiction: Assume that a certain self-similar scheme (1.2)-(1.4) satisfying Assumption 3.3 fails to converge. We need to show that there exists a sequence of numerical solutions $\{u^k\}$ of the scheme that harbors an ATES. This highly nontrivial task is fulfilled in three steps, which extend from §6 to §8. First, in §6 we develop a new technique, the extremum tracking. We introduce the notions of extremum paths and approximate extremum paths of a numerical solution of the aforementioned scheme. It turns out that a sequence of approximate extremum paths associated with a sequence of numerical solutions in $\widehat{\Psi}_{w_-,w_+,B}$ is necessarily a sequence of $\{\varepsilon_k\}$-paths, a building block of the ATES. Next, in §7, we analyze the asymptotic waves of the sequences of numerical solutions in a nonempty $\widehat{\Psi}_{w_-,w_+,B}$. Roughly speaking, a wave of a numerical solution is the generic structure of the numerical solution between two (approximate) extremum paths, and an asymptotic wave is a sequence of waves of a sequence of numerical solutions. Using similarity transforms and selecting subsequences, we show that there exists a sequence in $\widehat{\Psi}_{w_-,w_+,B}$ such that in a compact domain, the transition regions and the boundaries of all sufficiently strong asymptotic waves converge to $x = st$, which is the path of the discontinuity of the limit W of $\widehat{\Psi}_{w_-,w_+,B}$. Hence, these asymptotic waves must be ATWs. We then split these ATWs into ATDs. Finally, in §8, using Theorem 3.2 and Lemma 3.10, we complete the proof of Theorem 3.13 by showing that if W is a traveling expansion shock, then one of the ATDs must be an ATES.

6. EXTREMUM TRACKING

In this section we introduce and analyze the notions of extremum paths and approximate extremum paths of a numerical solution u. Theses paths will serve as the boundaries of the transition regions of the waves of the numerical solution.

We begin with the following simple observation:

Lemma 6.1. *If u is a uniformly bounded numerical solution of a scheme* (1.2)-(1.4), *then there are two positive constants C_1 and C_2 such that*

$$|\frac{du_j(t)}{dt}| \leq C_1/h \quad \text{and} \quad |\frac{du_j(t'')}{dt} - \frac{du_j(t')}{dt}| \leq (C_2/h^2)|t'' - t'|. \tag{6.1}$$

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Proof. Suppose that $|u|$ is bounded by the constant B, and that K is the Lipschitz coefficient of g: for any data $\{v_j\}_{j=1}^{2p}$ and $\{v'_j\}_{j=1}^{2p}$, we have

$$|g(v_1,\ldots,v_{2p}) - g(v'_1,\ldots,v'_{2p})| \le K\sum_{j=1}^{2p}|v_j - v'_j|.$$

Then

$$\left|\frac{du_j(t)}{dt}\right| = |\Delta_- g(u_{j-p+1}(t),\ldots,u_{j+p}(t))/h| \le K\sum_{i=-p+1}^{p}|\Delta_- u_{j+i}(t)|/h \le 4pKB/h = C_1/h,$$

and hence,

$$\begin{aligned}\left|\frac{du_j(t'')}{dt} - \frac{du_j(t')}{dt}\right| &= |\Delta_-(g(u_{j-p+1}(t''),\ldots,u_{j+p}(t'')) - g(u_{j-p+1}(t'),\ldots,u_{j+p}(t')))/h| \\ &\le 2K\sum_{i=j-p}^{j+p}|u_i(t'') - u_i(t')|/h \le 2K\sum_{i=j-p}^{j+p}\int_{t'}^{t''}(|\frac{du_i(t)}{dt}|/h)dt \\ &\le 2K(2p+1)|t''-t'|C_1/h^2 = (C_2/h^2)|t''-t'|.\end{aligned}$$

This completes the proof. □

I. *The extremum paths.* The set $X = \{x_j\}_{j=-\infty}^{\infty}$ is called the set of grid points and $L = X \times \mathbf{R}_+$ the set of grid lines. The solution u is defined on L. A finite set of successive grid points $\{x_p,\ldots,x_q\}$ with $q \ge p$ is said to be the *stencil of a spatial maximum*, or simply *an M-stencil* of u at the time t, provided $u_p(t) = \cdots = u_q(t)$, $u_{p-1}(t) < u_p(t)$, and $u_{q+1}(t) < u_q(t)$. Notions of *N-stencils* for minima and *E-stencils* for general extrema are defined similarly.

Lemma 6.2. *Suppose $\{x_p,\ldots,x_q\}$, $q \ge p$, is an M-stencil of u at the time t_0. Then there exists a $\delta > 0$ such that, when $|t-t_0| < \delta$, there exists at least one M-stencil $\{x_{p'},\ldots,x_{q'}\}$ of $u_j(t)$ at the time t that is a subset of $\{x_p,\ldots,x_q\}$.*

Proof. Since the numerical solution of a semidiscrete scheme is continuous in t, there exists a $\delta > 0$ such that for each t with $|t-t_0| < \delta$,

$$\max\{u_{p-1}(t), u_{q+1}(t)\} < \min\{u_p(t),\ldots,u_q(t)\}.$$

Since $\{x_p,\ldots,x_q\}$ is a finite set, it contains a subset $\{x_{p'},\ldots,x_{q'}\}$ with $u_{p'}(t) = \cdots = u_{q'}(t) = \max\{u_p(t),\ldots,u_q(t)\}$, $u_{p'-1}(t) < u_{p'}(t)$ and $u_{q'+1}(t) < u_{q'}(t)$. □

Motivated by Lemma 6.2, we define a ridge of u as follows:

Definition 6.3. A nonempty subset of L denoted by M_{t_1,t_2} is called *a ridge of the numerical solution u from t_1 to t_2* if

(i) For all $t \in [t_1, t_2]$ the set

$$P_M(t) \stackrel{\text{def}}{=} \{x_j : (x_j, t) \in M_{t_1,t_2}\} = \{x_{p(t)},\ldots,x_{q(t)}\}$$

is not empty, and is an M-stencil of u at t.

(ii) M_{t_1,t_2} satisfies the following "connectivity" condition: for each t on $[t_1,t_2]$, there exists a neighborhood $U(t)$ of t such that if $t' \in U(t)\cap[t_1,t_2]$, then $P_M(t') \subseteq P_M(t)$.

The set $P_M(t)$ is called *the x-projection of M_{t_1,t_2}* at t. The value of u along the ridge is denoted by $V_M(t)$: $V_M(t) = u_j(t)$ for $p(t) \le j \le q(t)$.

Hereafter , we use the notations M_{t_1,t_2}, $P_M(t)$ and $V_M(t)$ exclusively for the above notions. If for all $t \in [t_1,t_2]$ the M-stencil in (ii) is replaced by an N-stencil, then the set is called a *trough of u from t_1 to t_2*, and is denoted by N_{t_1,t_2}. The related notions $P_N(t)$ and $V_N(t)$ are defined similarly. Ridges and troughs are also called *extremum paths*. When we do not distinguish between ridges and troughs, we use E_{t_1,t_2}, $P_E(t)$ and $V_E(t)$ for either type. We add superscripts on M, N, or E to indicate several paths in one solution, sequences of paths associated with a sequence of solutions, or both. We make the convention that $E^1_{t_1,t_2} < (\le) E^2_{t_1,t_2}$ when $\max P_{E^1}(t) < (\le) \max P_{E^2}(t)$ for all $t \in [t_1,t_2]$. Let $t'' > t' \ge 0$. We say that a given E-stencil of u at t'' can be traced back to t' if it is the x-projection of a path E_{t_1,t_2} at t''.

Intuitively, if the scheme is nonoscillatory, no new extremum emerges for $t > 0$, and one should be able to trace any given E-stencil back to the initial time $t = 0$. This is indeed true (see Lemma 6.11) provided that the scheme satisfies the following TVD condition, also due to Tadmor [25].

Assumption 6.4 (Local Maximum Principle). If x_i belongs to an M(N)-stencil of $u_j(t)$ at the time t, then

$$\frac{du_i(t)}{dt} \le (\ge) 0.$$

Clearly, a scheme satisfying Assumption 3.3 also satisfies Assumption 6.4.

The main result of this section is the Backward Traceability Lemma 6.11. A string of lemmas are needed in its proof: the Local Nonoscillatory Property Lemmas 6.5 and 6.6, the Monotonicity Property Lemmas 6.7 and 6.8, the Sweeping Over Property Lemma 6.9 and the Order Preserving Property Lemma 6.10. The proof of Lemma 6.10 also needs Lemma 6.5. Lemmas 6.7, 6.9, 6.10 and 6.11 will play active roles in §7 and §8 in the arguments of wave separations, concentrations and splitting, and in the eventual proof of Theorem 3.13.

We now establish the two local nonoscillatory properties.

Lemma 6.5 (Local Nonoscillatory Property 1). *With Assumption 6.3, if $\{x_p,\ldots,x_q\}$ is an M-stencil of u at some t_0, then there exists a $\delta > 0$ such that at any $t \in [t_0, t_0+\delta)$, there is no N-stencil of u that is a subset of $\{x_p,\ldots,x_q\}$.*

Proof. Since u is continuous in t and $\{x_p,\ldots,x_q\}$ is an M-stencil of u at t_0, there exists a $\delta > 0$ such that for $t \in [t_0, t_0+\delta)$,

$$\max(u_{p-1}(t), u_{q+1}(t)) < \min_{p\le j\le q} u_j(t).$$

If $q \le p+1$, the lemma holds trivially. Otherwise, for any j with $p < j < q$, set

$$D_j(t) \stackrel{\text{def}}{=} \min\left(\max_{p\le l<j}(u_l(t) - u_j(t))_+, \max_{j<r\le q}(u_r(t) - u_j(t))_+\right)$$

and

$$D(t) \stackrel{\text{def}}{=} \max_{p<j<q} D_j(t).$$

Here, $(a)_+ = a$ for $a > 0$, and $(a)_+ = 0$ otherwise. Clearly, $D(t) \ge 0$, and $D(t) = 0$ if and only if there is no N-stencil of u at t that is a subset of $\{x_p, \dots, x_q\}$.

The function $D(t)$ is clearly continuous. Therefore, $\{t : D(t) > 0\} \cap [t_0, t_0+\delta)$ is an open set which is a union of disjoint open intervals: $\cup_\nu I_\nu$. It suffices to show that $D(t)$ is diminishing in I_ν for each ν. Fix an arbitrary $t \in I_\nu$. There exist an i with $p < i < q$, an l with $p \le l < i$, and an r with $i < r \le q$ such that $D(t) = \min(u_l(t) - u_i(t), u_r(t) - u_i(t))$. Clearly, x_l and x_r each belongs to an M-stencil of u, and x_i belongs to an N-stencil of u at t. Hence, Assumption 6.4 implies

$$\frac{du_l(t)}{dt} \le 0, \quad \frac{du_r(t)}{dt} \le 0, \quad \text{and} \quad \frac{du_i(t)}{dt} \ge 0.$$

By Lemma 6.1, for any $\varepsilon > 0$, if $t' \in I_\nu$ and $0 < t - t' < h^2\varepsilon/2C_2$, then

$$\frac{du_l(s)}{ds} \le \frac{\varepsilon}{2}, \quad \frac{du_r(s)}{ds} \le \frac{\varepsilon}{2}, \quad \text{and} \quad \frac{du_i(s)}{ds} \ge -\frac{\varepsilon}{2}$$

for all $s \in [t', t]$. It follows that

$$\begin{aligned} u_l(t') &\ge u_l(t) - \tfrac{\varepsilon}{2}(t - t'), \\ u_r(t') &\ge u_r(t) - \tfrac{\varepsilon}{2}(t - t'), \\ u_i(t') &\le u_i(t) + \tfrac{\varepsilon}{2}(t - t'). \end{aligned}$$

Hence,

$$\begin{aligned} D(t') &\ge D_i(t') \ge \min(u_l(t') - u_i(t'), u_r(t') - u_i(t')) \\ &\ge \min(u_l(t) - u_i(t), u_r(t) - u_i(t)) - \varepsilon(t - t') \\ &= D(t) - \varepsilon(t - t'). \end{aligned}$$

This implies $D(s'') - D(s') \ge -\varepsilon(s' - s'')$, for any $s', s'' \in I_\nu$ with $s' > s''$. The arbitrariness of ε then yields $D(s'') - D(s') \ge 0$. □

Lemma 6.6 (Local Nonoscillatory Property 2). *With Assumption 6.4, if*

$$u_{p-1}(t_0) > u_p(t_0) \ge \cdots \ge u_q(t_0) > u_{q+1}(t_0),$$

then there exists a $\delta > 0$ such that for $t \in [t_0, t_0+\delta)$

$$u_{p-1}(t) > u_p(t) \ge \cdots \ge u_q(t) > u_{q+1}(t). \tag{6.2}$$

Proof. Since the numerical solution is continuous in t, there exists a $\delta > 0$ such that

$$u_{p-1}(t) > \max_{p\le i\le q} u_i(t) \ge \min_{p\le i\le q} u_i(t) > u_{q+1}(t)$$

for $t \in [t_0, t_0+\delta)$. Define

$$D(t) = \max_{p\le l<q}\left(\max_{l<r\le q}(u_r(t) - u_l(t))_+\right).$$

Clearly, $D(t) \ge 0$, and $D(t) = 0$ if and only if (6.2) holds. It therefore suffices to show that $D(t)$ diminishes in $t \in [t_0, t_0+\delta)$. We omit the rest of the proof, which is similar to that for Lemma 6.5. □

Lemma 6.7 (First Monotonicity Property). *With Assumption 6.4, if M_{t_1,t_2} is a ridge, then the corresponding $V_M(t)$ is a decreasing function on $[t_1, t_2]$.*

Proof. Fix an arbitrary $\varepsilon > 0$. Let $t_1 < t' < t'' \le t_2$. By the "connectivity condition," for any $t \in [t', t'']$, there is a $\delta(t) \in (0, h^2\varepsilon/C_2)$ such that the interval $O_t = (t - \delta(t), t + \delta(t))$ satisfies

$$O_t \subseteq (t', t'') \quad \text{if} \quad t \in (t', t''). \tag{6.3}$$

$$P_M(s) \subseteq P_M(t) \quad \text{for} \quad s \in O_t. \tag{6.4}$$

By Assumption 6.4 and Lemma 6.1 we also have

$$\frac{du_j(s)}{ds} \le \varepsilon \quad \text{for} \quad s \in O_t \quad \text{and} \quad x_j \in P_M(t). \tag{6.5}$$

Since

$$\bigcup_{t\in[t',t'']} (t - \tfrac{1}{2}\delta(t), t + \tfrac{1}{2}\delta(t)) \supset [t', t''],$$

there exists a finite partition of $[t', t'']$: $t' = \tau_0 < \tau_1 < \cdots < \tau_n = t''$ such that

$$\bigcup_{\nu=0}^{n} (\tau_\nu - \tfrac{1}{2}\delta(\tau_\nu), \tau_\nu + \tfrac{1}{2}\delta(\tau_\nu)) \supset [t', t''].$$

Notice that the condition (6.3) implies that the inclusion of $(\tau_0 - \frac{1}{2}\delta(\tau_0), \tau_0 + \frac{1}{2}\delta(\tau_0))$ and $(\tau_n - \frac{1}{2}\delta(\tau_n), \tau_n + \frac{1}{2}\delta(\tau_n))$ is necessary to form the covering. Obviously, we may assume that

$$\frac{\delta(\tau_\nu) + \delta(\tau_{\nu+1})}{2} > \tau_{\nu+1} - \tau_\nu \quad \text{for each} \quad \nu = 0, 1, \dots, n-1.$$

Hence, either $\delta(\tau_{\nu+1}) > \tau_{\nu+1} - \tau_\nu$ or $\delta(\tau_\nu) > \tau_{\nu+1} - \tau_\nu$. In both cases, (6.4) and (6.5) imply

$$V_M(\tau_{\nu+1}) - V_M(\tau_\nu) \le \varepsilon(\tau_{\nu+1} - \tau_\nu),$$

which yields

$$V_M(t'') - V_M(t') \le \varepsilon(t'' - t').$$

It follows that

$$V_M(t'') \le V_M(t')$$

since ε is arbitrary. □

Lemma 6.8 (Second Monotonicity Property). *With Assumption 6.4, if M_{t_1,t_2} is a ridge of u with*

$$P_M(t) = \{x_{p(t)}, \ldots, x_{q(t)}\} \qquad \textit{for each} \quad t \in [t_1, t_2],$$

then

$$R(t) = \inf_{q(t)<j<\infty} u_j(t)$$

is an increasing function on $[t_1, t_2]$.

Proof. For any $\varepsilon > 0$, let $t_1 \le t' < t'' \le t_2$, $t'' - t' < \frac{h^2\varepsilon}{2C_2}$. Suppose $\{j_\nu\}_{\nu=1}^{\infty}$ satisfy $j_\nu > q(t'')$ and

$$\lim_{\nu\to\infty} u_{j_\nu}(t'') = R(t'').$$

Then Assumption 6.4 and the continuity of the numerical flux imply

$$\liminf_{\nu\to\infty} \frac{du_{j_\nu}(t'')}{dt} \ge 0,$$

which, by Lemma 6.1, implies

$$\liminf_{\nu\to\infty} \frac{du_{j_\nu}(t)}{dt} \ge -\frac{\varepsilon}{2}$$

uniformly for $t \in [t', t'']$. Therefore,

$$\begin{aligned} R(t') &= \inf_{q(t')<j<\infty} u_j(t') \le \liminf_{\nu\to\infty} u_{j_\nu}(t') \\ &\le \lim_{\nu\to\infty} u_{j_\nu}(t'') + \varepsilon(t'' - t') \\ &= R(t'') + \varepsilon(t'' - t'). \end{aligned}$$

Now, for any $s', s'' \in [t_1, t_2]$ with $s' < s''$, we partition $[s', s'']$ into a finite number of disjoint subintervals of lengths bounded by $h^2\varepsilon/2C_2$, we apply the above inequality to each of the subintervals, and we sum the results up. This yields $R(s') \le R(s'') + \varepsilon(s'' - s')$. Therefore, $R(s') \le R(s'')$ since ε is arbitrary. □

The following lemma implies that along its route of propagation, an extremum path of u sweeps over all the grid points in between.

Lemma 6.9 (Sweeping Over Property). *With the Assumption 6.4, if M_{t_1,t_2} is a ridge of u such that*

$$x_p = \min[P_M(t_1) \textstyle\bigcup P_M(t_2)], \tag{6.6}$$

and

$$x_q = \max[P_M(t_1) \textstyle\bigcup P_M(t_2)], \tag{6.7}$$

then

$$\bigcup_{t_1 \le t \le t_2} P_M(t) \supseteq \{x_j : p \le j \le q\}. \tag{6.8}$$

Proof. Assume j is an integer between p and q such that

$$x_j \notin \bigcup_{t_1 \le t \le t_2} P_M(t).$$

Then (6.6) and (6.7) imply that both

$$Q_l = \{t \in [t_1, t_2] : \max P_M(t) < x_j\}$$

and

$$Q_r = \{t \in [t_1, t_2] : \min P_M(t) > x_j\}$$

are not empty. Clearly, $Q_l \bigcup Q_r = [t_1, t_2]$ and $Q_l \bigcap Q_r = \emptyset$. These contradict the connectivity of the interval $[t_1, t_2]$ since both Q_l and Q_r are open in $[t_1, t_2]$ by the "connectivity" of M_{t_1,t_2}. Therefore, (6.8) holds. □

The next lemma implies that the forward extremum path of u starting from an E-stencil at any time t_0 is unique.

Lemma 6.10 (Order Preserving Property). *Let $E^{(1)}_{t_1,t_2}$ and $E^{(2)}_{t_1,t_2}$ with $t_1 < t_2$ be two extremum paths of u. If*

$$\max P_{E^{(1)}}(t_2) < \min P_{E^{(2)}}(t_2),$$

then $E^{(1)}_{t_1,t_2} < E^{(2)}_{t_1,t_2}$.

Proof. By the "connectivity condition," the set $\{t \in [t_1, t_2] : \max P_{E^{(1)}}(t) < \min P_{E^{(2)}}(t)\}$ is a nonempty open subset of $[t_1, t_2]$. If this subset is not $[t_1, t_2]$ itself, then there is a $t' \in [t_1, t_2]$ such that $\max P_{E^{(1)}}(t) < \min P_{E^{(2)}}(t)$ holds for all $t \in (t', t_2]$, and either $P_{E^{(1)}}(t') = P_{E^{(2)}}(t')$ or $\max P_{E^{(2)}}(t') < \min P_{E^{(1)}}(t')$. The former case contradicts Lemma 6.5, and the latter contradicts the "connectivity condition." □

We are ready to present and prove our main result of this section.

Lemma 6.11 (Backward Traceability Property). *With the Assumption 6.4, any M-stencil $\{x_p, \ldots, x_q\}$ at any $T > 0$ can be traced back to $t = 0$.*

Proof. Let $\mathcal{T}$ be the set of t for which there exists an $M_{t,T}$ with $P_M(T) = \{x_p, x_{p+1}, \ldots, x_q\}$. We prove the lemma by showing that $\mathcal{T}$ is nonempty and is both open and closed in $[0, T]$.

"Open": Let $t' \in \mathcal{T}$. Then there exists an $M_{t',T}$ such that $P_M(T) = \{x_p, x_{p+1}, \ldots, x_q\}$. As we have seen repeatedly, there exists a $\delta_1 > 0$ such that for $t \in [t' - \delta_1, t')$

$$\max(u_{p(t')-1}(t), u_{q(t')+1}(t)) < \min_{p(t') \le j \le q(t')} u_j(t). \tag{6.9}$$

Let $N(t)$ be the number of the M-stencils $\{x_{p_i(t)},\dots,x_{q_i(t)}\} \subset P_M(t')$ at the time t, $i = 1,\dots,N(t)$. The function $N(t)$ is defined on $[t'-\delta_1,t')$. The range of $N(t)$ is a finite set of integers $\{n_\nu\}_{\nu=1}^m$. Let $n_1 > n_2 > \cdots > n_m$. We claim that there exists a $\delta \in (0,\delta_1]$ such that $N(t) \equiv n_m$ for $t \in [t'-\delta,t')$. Indeed, the claim is trivially true for $m = 1$. In the case $m > 1$, consider the set $\{t \in [t'-\delta_1,t') : N(t) > n_m\}$. Lemma 6.2 implies that this set is open in $[t'-\delta_1,t')$. If the claim were false, there would be a nonempty open interval $(s_1,s_2) \subset [t'-\delta_1,t')$ such that $N(s_1) = n_m$ and $N(t) > n_m$ for $t \in (s_1,s_2)$. This would contradict Lemmas 6.5 and 6.6. Hence, the claim is true.

Now for any fixed $t \in [t'-\delta,t')$, let $\{P^{(i)}(t)\}_{i=1}^{n_m}$ be the collection of M-stencils in order of increasing space coordinates. Fix any i between 1 and n_m. Then for any $t \in (t'-\delta,t')$, define $M_{t,T}$ by letting $M_{t',T} \subset M_{t,T}$ and $P_M(s) = P^{(i)}(s)$ for $s \in [t,t')$. To see that the $M_{t,T}$ thus defined is a ridge of u, it suffices to verify the "connectivity condition". Indeed, since $N(t) \equiv n_m$ for $t \in [t'-\delta,t')$, Lemma 6.2 implies that for any fixed $t \in (t'-\delta,t')$, there exists a $\delta_2 > 0$ such that $P^{(i)}(s) \subseteq P^{(i)}(t)$ for any i between 1 and n_m, provided that $|s-t| < \delta_2$. This shows that T is open in $[0,T]$.

"Closed": Suppose that $\{\tau_\mu\}_{\mu=0}^\infty$ is a decreasing sequence of real numbers such that $\tau_0 = T$ and $\tau_\mu \to t' \in [0,T)$ as $\mu \to \infty$. Suppose also that there exists a sequence of ridges $\{M^\mu_{\tau_\mu,T}\}$ such that $P_{M^\mu}(T) = \{x_p, x_{p+1},\dots,x_q\}$ and $P_{M^\mu}(t) = \{x_{p^\mu(t)},\dots,x_{q^\mu(t)}\}$ when $\mu = 1,2,3,\dots$ and $\tau_\mu \le t \le T$.

We claim that $x_{p^\mu(t)}$ and $x_{q^\mu(t)}$ are uniformly bounded. Let us first show that there exists a number $X_{\max}$ such that for all μ, $x_{q^\mu(t)} < X_{\max}$ when $\tau_\mu \le t \le T$.

Noticing that $P_{M^\mu}(T)$ is independent of μ, we set $a \overset{\text{def}}{=} V_{M^\mu}(T)$ and $\sigma \overset{\text{def}}{=} [V_{M^\mu}(T) - R_{M^\mu}(T)]/3$, where $R_{M^\mu}(t) = \inf_{j \ge q^\mu(t)} u_j(t)$. By Lemmas 6.7 and 6.8, for all $t \in [\tau_\mu, T]$,

$$V_{M^\mu}(t) \ge a \quad \text{and} \quad R_{M^\mu}(t) \le a - 3\sigma. \tag{6.10}$$

Suppose that l is a positive integer such that $\Delta t \overset{\text{def}}{=} (T-t')/l < \sigma h/C_1$, where C_1 is the positive constant in Lemma 6.1. We show that for each $k = 0,\dots,l-1$, there is a constant $X^k_{\max}$ such that for all $t \in [T-(k+1)\Delta t, T-k\Delta t]$ and $t \ge \tau_\mu$, we have $x_{q^\mu(t)} < X^k_{\max}$. We apply induction to k. By (6.10), there is an integer $j_0 > q$ such that $u_{j_0}(T) < a - 2\sigma$. Hence, by Lemma 6.1, for all $t \in [T-\Delta t, T]$ there holds $u_{j_0}(t) < a - \sigma$. Therefore, for all $t \in [T-\Delta t, T]$ we have $q^\mu(t) < j_0$ by the First Monotonicity Property Lemma 6.7, the Sweeping Over Property Lemma 6.9, and the first inequality in (6.10). Hence, $X^0_{\max} \overset{\text{def}}{=} x_{j_0}$ has the desired property. Similarly, if $X^{k-1}_{\max}$ has the desired property, then there is a $j_k > \max_\mu q^\mu(T-k\Delta t)$ such that for all $t \in [T-(k+1)\Delta t, T-k\Delta t]$ one has $u_{j_k}(t) < a - \sigma$. Therefore, $X^k_{\max} = x_{j_k}$ has the desired property. Now $X_{\max} = \max_{k=0,\dots,l-1} X^k_{\max}$ is an upper bound of $x_{q^\mu(t)}$. Similarly, there exists a number $X_{\min}$ that is a lower bound of $x_{p^\mu(t)}$. This proves the claim.

We show that there is a sequence of ridges $\{M_{\tau_\mu,T}\}_{\mu=1}^\infty$ such that $P_M(T) = \{x_p, x_{p+1},\dots,x_q\}$ and

$$M_{\tau_{\mu_1},T} \subset M_{\tau_{\mu_2},T} \quad \text{if} \quad \mu_1 < \mu_2. \tag{6.11}$$

Because of (6.11), it is not necessary to use superscripts to distinguish different ridges in this sequence.

To see this, let us assume that $\{P^\mu_i\}_{i=1}^{m_\mu}$ is the collection of M-stencils of u at τ_μ between $X_{\min}$ and $X_{\max}$. Since $x_{p^\mu(t)}$ and $x_{q^\mu(t)}$ are uniformly bounded, m_μ is a bounded sequence. Inductively, we select an M-stencil $P^\mu = P^\mu_{i_\mu}$ at τ_μ for each natural number μ as follows: We are able to select a $P^1 = P^1_{i_1}$ such that it is the x-projections at τ_1 of an infinite subsequence $(SQ)_1$ of $\{M^\mu_{\tau_\mu,T}\}_{\mu=1}^\infty$ because m_1 is bounded. In general, if $(SQ)_\mu$ and P^μ have been selected, we are able to select a $P^{\mu+1} = P^{\mu+1}_{i_{\mu+1}}$ such that it is the x-projections at $\tau_{\mu+1}$ of an infinite subsequence $(SQ)_{\mu+1}$ of $(SQ)_\mu$ because $\{m_\mu\}_{\mu=1}^\infty$ is a bounded sequence. Thus, P^μ is selected for every natural number μ. Now we define $M_{\tau_\mu,T}$ to be the unique ridge of u such that $P_M(\tau_\mu) = P^\mu$ (uniqueness is guaranteed by Lemma 6.10). The relation (6.11) follows from the uniqueness.

Suppose that $P_M(t) = \{x_{p(t)},\dots,x_{q(t)}\}$ and that

$$\tilde p = \lim_{\mu\to\infty} \inf p(\tau_\mu), \quad \text{and} \quad \tilde q = \lim_{\mu\to\infty} \sup q(\tau_\mu).$$

We have $\tilde p \le \tilde q$. Since $p(\tau_\mu)$ and $q(\tau_\mu)$ are uniformly bounded, $q(\tau_\mu) \le \tilde q$ and $p(\tau_\mu) \ge \tilde p$ for sufficiently large μ. Moreover, Lemmas 5.6 and 5.9 and the continuity of u in t imply that

$$u_{\tilde p}(t') = \cdots = u_{\tilde q}(t') = \lim_{\mu\to\infty} V(\tau_\mu).$$

Let $\hat p$ and $\hat q$ be the integers such that $\hat p \le \tilde p \le \tilde q \le \hat q$ with

$$u_{\hat p}(t') = \cdots = u_{\hat q}(t'),$$

and

$$u_{\hat p-1}(t') \ne u_{\hat p}(t'), \quad u_{\hat q+1}(t') \ne u_{\hat q}(t').$$

To see that such a $\hat q$ does exist, recall that we have indeed shown the existence of an $x_{j_{l-1}} = X^{l-1}_{\max} > \tilde q$ such that $u_{j_{l-1}}(t') < a - \sigma$. Similarly, $\hat p$ exists. The collection $\{x_{\hat p},\dots,x_{\hat q}\}$ is an M-stencil of u at t', because otherwise, by Lemmas 6.5 and 6.6, $\{x_{\hat p-1},\dots,x_{\hat q+1}\}$ would not contain any M-stencil at $t = \tau_\mu$ for sufficiently large μ and the existence of the aforementioned $M_{\tau_\mu,T}$ would be violated. Set $P_M(t') = \{x_{\hat p},\dots,x_{\hat q}\}$. Then $M_{t',T}$ is well defined. This shows that T is closed in $[0,T]$. □

The extremum paths of u are closely related to the spatial variation of u.

As usual, the total spatial variation of a numerical solution u at t is defined by

$$TV_u(t) = \sum_{j=-\infty}^{\infty} |u_j(t) - u_{j-1}(t)|.$$

Under Assumption 6.4, if $TV_u(0)$ is bounded, then $TV_u(t) \le TV_u(0)$ and both

$$V^L = \lim_{j\to-\infty} u_j(t) \quad \text{and} \quad V^R = \lim_{j\to\infty} u_j(t)$$

exist and are independent of t. If the x-projections $\{P_{E^m}(t)\}_{m=m'}^{m''}$ of a collection of extremum paths, $E_{0,t}^{m'} < \cdots < E_{0,t}^{m''}$, are all the $m'' - m' + 1$ distinct E-stencils of u at t with $\max P_{E^{-1}}(t) < 0 \le \max P_{E^0}(t)$, then clearly

$$TV_u(t) = |V^L - V_{E^{m'}}(t)| + \sum_{m=m'+1}^{m''} |V_{E^m}(t) - V_{E^{m-1}}(t)| + |V^R - V_{E^{m''}}(t)|. \tag{6.12}$$

In the case m' and/or m'' is infinity, the above formula still holds if the corresponding bound(s) of the summations is (are) replaced by $-\infty$ and/or ∞ , and if $|V^L - V_{M^{m'}}(t)|$ and/or $|V_R - V_{E^{m''}}(t)|$ vanish(es) accordingly.

Suppose $0 \le t < t'$. One sees easily from the formula (6.12) that the variation decay

$$DTV_u(t,t') \stackrel{\text{def}}{=} TV_u(t) - TV_u(t')$$

is twice the sum of the decreases (increases) of the values of u along all ridges (troughs), including the ones that vanish, when t increases from t' to t. This observation motivates the following simple yet useful result.

Lemma 6.12. *Suppose the numerical solution u of a scheme (1.2)-(1.4) satisfies Assumption 6.4. Suppose also that $E_{t,t'}^1 \le E_{t,t'}^2$ are two extremum paths of u such that either $P_{E^1}(t')$ and $P_{E^2}(t')$ are identical or they are two successive E-stencils of u at t'. Let*

$$u^M = \max_{\min P_{E^1}(t') \le x \le \max P_{E^2}(t')} u(x,t') \quad \text{and} \quad u^N = \min_{\min P_{E^1}(t') \le x \le \max P_{E^2}(t')} u(x,t'). \tag{6.13}$$

(i) *If $\min P_{E^1}(t) \le x_j \le \max P_{E^2}(t)$, then*

$$u^N - DTV_u(t,t')/2 \le u_j(t) \le u^M + DTV_u(t,t')/2.$$

(ii) *If $\min P_{E^1}(t) \le x_p < x_q \le \max P_{E^2}(t)$, and $u_q(t) - u_p(t)$ has the opposite sign of $V_{E^2}(t) - V_{E^1}(t)$, then*

$$|u_q(t) - u_p(t)| \le DTV_u(t,t')/2.$$

Proof. Without loss of generality, assume that $E_{t,t'}^1$ and $E_{t,t'}^2$ are a trough and a ridge, respectively. It follows that $u^M = V_{E^2}(t')$ and $u^N = V_{E^1}(t')$. Hence, using Lemma 6.7, we have

$$\begin{aligned} DTV_u(t,t') &\ge 2[(V_{E^2}(t) - V_{E^2}(t')) + (V_{E^1}(t') - V_{E^1}(t)) + (u_j(t) - V_{E^2}(t))] \\ &\ge 2(u_j(t) - V_{E^2}(t')) = 2(u_j(t) - u^M). \end{aligned}$$

This proves the second part of the desired inequality in (i). The first part can be similarly shown.

Next,

$$\begin{aligned} DTV_u(t,t') &\ge \sum_{j=\min P_{E^1}(t)}^{\max P_{E^2}(t)-1} |u_{j+1}(t) - u_j(t)| - \sum_{j=\min P_{E^1}(t')}^{\max P_{E^2}(t')-1} |u_{j+1}(t') - u_j(t')| \\ &= \sum_{j=\min P_{E^1}(t)}^{\max P_{E^2}(t)-1} |u_{j+1}(t) - u_j(t)| - (V_{E^2}(t') - V_{E^1}(t')) \\ &\ge \sum_{j=\min P_{E^1}(t)}^{\max P_{E^2}(t)-1} |u_{j+1}(t) - u_j(t)| - (V_{E^2}(t) - V_{E^1}(t)) \\ &\ge |V_{E^2}(t) - u_q(t)| + |u_q(t) - u_p(t)| + |u_p(t) - V_{E^1}(t)| - (V_{E^2}(t) - V_{E^1}(t)) \\ &\ge V_{E^2}(t) - u_q(t) + u_p(t) - u_q(t) + u_p(t) - V_{E^1}(t) - (V_{E^2}(t) - V_{E^1}(t)) \\ &= 2(u_p(t) - u_q(t)). \end{aligned}$$

This proves (ii). □

II. *The approximate extremum paths.* Given a path E_{t_1,t_2}, we would like, ideally, to have a function $x_E(t)$: $[t_1, t_2] \to \mathbf{R}$ that satisfies the following conditions:

(1) There is a finite partition of $[t_1, t_2]$

$$t_1 = \tau_0 < \tau_1 < \cdots < \tau_n = t_2$$

such that $x_E(t)$ is a constant on each subinterval $(\tau_{\nu-1}, \tau_\nu)$ for $1 \le \nu \le n$.

(2) $x_E(t) \in P_E(t)$ for $t \in [t_1, t_2]$.

It is not clear if such a function $x_E(t)$ exists. However, we do know that there exist piecewise constant functions of t that represent E_{t_1,t_2} sufficiently well. Without loss of generality, we discuss spatial maxima only.

Lemma 6.13. *For each ridge M_{t_1,t_2} of a numerical solution u generated by a scheme (1.2)-(1.4) satisfying Assumption 6.4, and for any $\varepsilon > 0$, there exists a piecewise constant function $x_M^\varepsilon(t) : [t_1, t_2] \to \mathbf{R}$ and a finite partition of $[t_1, t_2] : t_1 = \tau_0 < \tau_1 < \cdots < \tau_n = t_2$ such that $x_M(t) \equiv c_\nu$ and*

$$c_\nu \in P_M(\tau_{\nu-1}) \bigcap P_M(\tau_\nu) \tag{6.14}$$

on each subinterval $(\tau_{\nu-1}, \tau_\nu)$ for $1 \le \nu \le n$. Furthermore, at any $t \in [t_1, t_2]$, for any x_i between (and including) $x_M^\varepsilon(t)$ and $P_M(t)$, we have

$$|V_M(t) - u_i(t)| < \varepsilon. \tag{6.15}$$

Finally, for $V_M^\varepsilon(t) \stackrel{\text{def}}{=} u(x_M^\varepsilon(t), t)$,

$$TV_{t_1 \le t \le t_2}(V_M^\varepsilon(t)) < V_M(t_1) - V_M(t_2) + \varepsilon/2. \tag{6.16}$$

Proof. We repeat the construction of the finite covering in the proof of Lemma 6.7 with the following modifications: First we replace $[t', t'']$ by $[t_1, t_2]$. Second, for each $t \in [t_1, t_2]$ we choose the positive $\delta(t)$ so small that in addition to (6.3) and (6.4), the following inequalities also hold for $s \in O_t$ and $x_j \in P_M(t)$:

$$|u_j(s) - u_j(t)| < \frac{\varepsilon}{2} \tag{6.17}$$

and

$$\frac{du_j(s)}{ds} < \frac{\varepsilon}{4(t_2 - t_1)}, \tag{6.18}$$

where (6.18) can be achieved because of Assumption 6.4 and Lemma 6.1.

With the finite covering, for each $\nu = 1, \dots, n$, if $\delta(\tau_\nu) > \tau_\nu - \tau_{\nu-1}$, we choose c_ν to be any number in $P_M(\tau_{\nu-1})$, otherwise we choose c_ν to be any number in $P_M(\tau_\nu)$. Set

$$x^\varepsilon_M(t) \equiv c_\nu \quad \text{for} \quad \tau_{\nu-1} < t < \tau_\nu. \tag{6.19}$$

Now (6.14) follows from the "connectivity" condition of M_{t_1,t_2}, and (6.15) from (6.17). To show that (6.16) holds, notice that

$$I = TV_{t_1 \le t \le t_2}(V^\varepsilon_M(t)) = \sum_{\nu=1}^n \int_{\tau_{\nu-1}}^{\tau_\nu} \left|\frac{dV^\varepsilon_M(t)}{dt}\right| dt.$$

Suppose $x_{j_\nu} = c_\nu$; then

$$\begin{aligned} I &= \sum_{\nu=1}^n \int_{\tau_{\nu-1}}^{\tau_\nu} \left|\frac{du_{j_\nu}(t)}{dt}\right| dt = \sum_{\nu=1}^n \int_{\tau_{\nu-1}}^{\tau_\nu} \left(-\frac{du_{j_\nu}(t)}{dt} + 2\left(\frac{du_{j_\nu}(t)}{dt}\right)_+ \right) dt \\ &\le V_M(t_1) - V_M(t_2) + \frac{2\varepsilon}{4(t_2 - t_1)} \sum_{\nu=1}^n \int_{\tau_{\nu-1}}^{\tau_\nu} dt = V_M(t_1) - V_M(t_2) + \varepsilon/2. \end{aligned}$$

This completes the proof of the lemma. □

We call the function $x^\varepsilon_M(t)$ an *ε-ridge to* M_{t_1,t_2}. Similarly, we have the notion *ε-troughs*. Both ε-ridges and ε-troughs are called ε-E paths.

We have the following result concerning the notions of ε-E paths of this section, the ε-paths in the sense of Definition 3.5, and the sets $\widehat{\Psi}_{w_-,w_+,B}$ introduced in §3. Recall that $\{\varepsilon_k\}$ is a sequence of positive numbers such that $\varepsilon_k \to 0$ as $k \to \infty$.

Lemma 6.14. *Suppose that all the sequences of a set* $\widehat{\Psi}_{w_-,w_+,B}$ *consist of numerical solutions generated by the same scheme* (1.2)-(1.4) *satisfying Assumption* 6.4, *and that* $\{u^k\} \in \widehat{\Psi}_{w_-,w_+,B}$. *Let k by a positive integer. If* $x^{\varepsilon_k}_E(t)$ *is an* ε_k*-E path of* u^k, *then it is also an* ε_k*-path of the first type with respect to* u^k.

Proof. The conditions (i) and (ii) in Definition 3.5 are the same as the relation (6.14) in Lemma 6.13 where ε-E paths are defined. The relation (6.15) verifies the condition (iii) in Definition 3.5. Finally, suppose $x^{\varepsilon_k}_E(t)$ is an approximate path to the exact one $E^k_{0,1}$. By Lemma 6.12, the condition (iii) in the definition of $\widehat{\Psi}_{w_-,w_+,B}$ implies that $|V_{E^k}(0) - V_{E^k}(1)| < \varepsilon_k/2$. Combining this inequality and (6.16), one verifies the condition (iv) in Definition 3.5. □

We end this section by extending Lemma 6.12 to ε-E paths.

Lemma 6.15. *Suppose that, in addition to the conditions and the notations of Lemma* 6.12. $x_{I(t)}$ *and* $x_{J(t)}$ *are ε-E paths to* $E^1_{t,t'}$ *and* $E^2_{t,t'}$ *respectively, such that* $x_{I(t')} \in P_{E^1}(t')$ *and* $x_{J(t')} \in P_{E^2}(t')$.

(i) *If* $x_{I(t)} \le x_j \le x_{J(t)}$, *then*

$$u^N - DTV_u(t,t')/2 - \varepsilon \le u_j(t) \le u^M + DTV_u(t,t')/2 + \varepsilon.$$

(ii) *If* $x_{I(t)} \le x_p < x_q \le x_{J(t)}$, *and* $u_q(t) - u_p(t)$ *has the opposite sign of* $V_{E^2}(t) - V_{E^1}(t)$, *then*

$$|u_q(t) - u_p(t)| \le DTV_u(t,t')/2 + 2\varepsilon.$$

Proof. The inequalities follow from Lemma 6.12 and the relation (6.17) immediately. □

Remark. Lemmas 6.12 and 6.15 also hold when $u(x,t')$ is monotone between $-\infty$ and $\max P_{E^2}(t')$, or when $u(x,t')$ is monotone between $\min P_{E^1}(t)$ and ∞. In the former case, $\min P_{E^1}(t)$ is replaced by $-\infty$ and $V_{E^1}(t)$ is replaced by V^L; in the latter case, $\max P_{E^2}(t)$ is replaced by ∞ and $V_{E^2}(t)$ is replaced by V_R. The proof is similar to that in the standard case and is omitted.

7. WAVE SEPARATIION, CONCENTRATIONS AND SPLITTINGS

By Theorem 3.2, to prove Theorem 3.13, it suffices to show that if $\widehat{\Psi}_{w_-,w_+,B} \ne \emptyset$, and if its limit W is a traveling expansion shock, then it contains a sequence that harbors an ATES. In order to extract the ATES, we screen the sequence of $\widehat{\Psi}_{w_-,w_+,B}$ with the waves in the following domains:

$$\begin{aligned} \Omega &\stackrel{\text{def}}{=} \{(x,t): \quad |x - st| \le 1, \quad 0 \le t \le 1\}, \\ \Omega^c_\delta &\stackrel{\text{def}}{=} \{(x,t): |x - st| < \delta, \quad 0 \le t \le 1\}, \\ \Omega^-_\delta &\stackrel{\text{def}}{=} \{(x,t): \quad -1 \le x - st \le -\delta, \quad 0 \le t \le 1\}, \\ \Omega^+_\delta &\stackrel{\text{def}}{=} \{(x,t): \quad \delta \le x - st \le 1, \quad 0 \le t \le 1\}, \end{aligned}$$

where $x = st$ is the path of the discontinuity of W, and δ is an arbitrary constant in $(0,1)$.

We first collect a few facts concerning similarity transforms and the set $\widehat{\Psi}_{w_-,w_+,B}$.

We require that the mapping $T^\gamma_{x',t'}$ preserve the indices of the schemes. Namely, if $u = T^\gamma_{x',t'}\tilde{u}$, then $u_j(t) = \tilde{u}_j(t' + \gamma t)$, i.e., $\tilde{u}_j(t) = u_j((t - t')/\gamma)$.

Lemma 7.1. *Suppose that* $\{\tilde{x}^k(t), \tilde{y}^k(t)\}_{k=1}^\infty$ *is an ATW of* $\{\tilde{u}^k\}_{k=1}^\infty \in \widehat{\Psi}_{w_-,w_+,B}$, *with the limit path* $x = \alpha t + \beta$ *and the two states L and R. Let γ be a constant in* $(0,1)$, t' *and* x' *be two* *constants such that $0 \le t' \le 1-\gamma$ and that $x' = st'$. For each k, let $\bar{z}^k(t)$ be an ε_k-E path with respect to $\bar{u}^k$ from t' to t^k where $t'+\gamma \le t^k \le 1$. Suppose also that $\bar{x}^k(t) < \bar{z}^k(t) < \bar{y}^k(t)$ when $t' \le t \le t^k$. We then have:*

(i) $T^{\gamma}_{x',t'}\hat{\Psi}_{w_-,w_+,B} \subset \hat{\Psi}_{w_-,w_+,B}$.

(ii) *$\{x^k(t), y^k(t)\}_{k=1}^{\infty}$ is an ATW of $\{u^k\}_{k=1}^{\infty}$ with the limit path $x = \alpha t + \beta'$ and the two states L and R. Here, $u^k = T^{\gamma}_{x',t'}\bar{u}^k$, $x^k(t) = [\bar{x}^k(t'+\gamma t) - x']/\gamma$, $y^k(t) = [\bar{y}^k(t'+\gamma t) - x']/\gamma$, and $\beta' = (\alpha t' + \beta - x')/\gamma$.*

(iii) *If $\lim_{k\to\infty} \bar{u}^k(\bar{z}^k(t^k), t^k) = \tilde{C}$ exists, and $\tilde{C} \ne L$, then $\{\bar{x}^k(t), \bar{z}^k(t)\}_{k=1}^{\infty}$ is also an ATW of $\{u^k\}_{k=1}^{\infty}$ with the limit path $x = \alpha t + \beta'$ and the two states L and $\tilde{C}$, where $z^k(t) = [\bar{z}^k(t'+\gamma t) - x']/\gamma$.*

Proof. Notice that $x^k(t)$, $y^k(t)$ and $z^k(t)$ are the $S^{\gamma}_{x',t'}$ image of $\bar{x}^k(t)$, $\bar{y}^k(t)$ and $\bar{z}^k(t)$, respectively, and the line $x = \alpha t + \beta'$ is the $S^{\gamma}_{x',t'}$ image of the line $x = \alpha t + \beta$. Since a scaling and a displacement of the independent variables are the only effects of the mapping $T^{\gamma}_{x',t'}$ on the the numerical solutions, (i) and (ii) are trivially true. To verify (iii), it suffices to notice that by Lemma 6.14, $z^k(t)$ is an ε_k-path of the first type, which together with $\lim_{k\to\infty} \bar{u}^k(\bar{z}^k(t^k), t^k) = \tilde{C}$ imply $u^k(z^k(t), t) \to \tilde{C}$ uniformly for $0 \le t \le 1$. □

We now state the first main result of this section.

Lemma 7.2 (Wave Separation and Concentration). *If $\hat{\Psi}_{w_-,w_+,B} \ne \emptyset$, then it contains a sequence $\{u^k\}$ such that*

$$|u^k(x,t) - W(x,t)| < \varepsilon_k \quad for \quad (x,t) \in \Omega^-_{\varepsilon_k} \bigcup \Omega^+_{\varepsilon_k}.$$

See the Appendix for a quite involved proof of the lemma.

Letting $\Psi_{w_-,w_+,B}$ be the subset of $\hat{\Psi}_{w_-,w_+,B}$ that consists of all the sequences satisfying the conclusion of Lemma 7.2, we can restate the lemma as: If $\hat{\Psi}_{w_-,w_+,B} \ne \emptyset$, then $\Psi_{w_-,w_+,B} \ne \emptyset$.

It makes sense to call Lemma 7.2 wave separation and concentration because if $\{x^k(t), y^k(t)\}$ is a ATW of $\{u^k\} \in \Psi_{w_-,w_+,B}$ with the limit path $x = \alpha t + \beta$, then it is either separated from $x = st$ in the sense that $|\alpha t + \beta - st| \ge 1$ for $0 \le t \le 1$, or it is concentrated on $x = st$ in the sense that $\alpha t + \beta = st$ for $0 \le t \le 1$.

Next, we use Lemma 6.11 to split the concentrated ATWs backwardly into ATDs and other insignificant waves. Assumption 6.4 is adequate for most arguments. However, we do need Assumption 3.3 to establish the essential monotonicity property of the waves.

Here is the second main result of this section.

Lemma 7.3. *Suppose that the numerical scheme satisfies Assumption 3.3. Then for any $\rho > 0$, if $\Psi_{w_-,w_+,B}$ is not empty, it contains a sequence $\{u^k(x,t)\}$ that has $\tilde{N}(\rho)$ ATWs*

$$\{x^k_{I^k_i(t)}, x^k_{J^k_i(t)}\}_{k=1}^{\infty}, \quad i = 1, \ldots, \tilde{N}(\rho)$$

with a common limit path $x = st$, where $\tilde{N}(\rho) = O(1/\rho)$. Moreover:

(i) *For $i = 1, \ldots, \tilde{N}(\rho)$, if $\tilde{L}_i$ and $\tilde{R}_i$ are the two states of the ith ATW, then*

$$\tilde{L}_i, \tilde{R}_i \in \{w : f(w) = f[w; w_-, w_+]\}.$$

(ii) *If $st - 1 \le x^k_j < x^k_{j+1} \le x^k_{I^k_1(t)}$ or $x^k_{J^k_{\tilde{N}(\rho)}(t)} \le x^k_j < x^k_{j+1} \le st + 1$, and $0 \le t \le 1$, then $|u^k_{j+1}(t) - u^k_j(t)| \le \rho + 2\varepsilon_k$.*

(iii) *If $J^k_{i-1}(t) \le j < j+1 \le I^k_i(t)$, and $0 \le t \le 1$, then $|u^k_{j+1}(t) - u^k_j(t)| \le \rho + \varepsilon_k$.*

Furthermore, all the $\tilde{N}(\rho)$ ATWs are ATDs.

Proof. During the proof we may repeatedly select subsequences and construct new sequences of $\Psi_{w_-,w_+,B}$. To avoid complicated notation, the sequences will be denoted by $\{u^k\}$ unless otherwise specified.

We start at $t = 1$. For each k, a *complete region of monotonicity* (CRM) of u^k at $t = 1$, denoted by $\tilde{M}^k$, is a set of consecutive grid points $\tilde{M}^k = \{x^k_{p^k}, \ldots, x^k_{q^k}\}$ such that

(1) $\tilde{M}^k \subset [s-1, s+1]$.

(2) $u^k_j(1)$ is monotone with respect to j when $p^k \le j \le q^k$.

(3) If either $x^k_{p^k-1}$ or $x^k_{q^k+1}$ is added to $\tilde{M}^k$, (1) or (2) no longer holds.

Set $\tilde{L}^k = u^k_{p^k}(1)$, $\tilde{R}^k = u^k_{q^k}(1)$ and $G^k = \tilde{R}^k - \tilde{L}^k$. We use subscripts to distinguish several CRMs and related notions. Fix an arbitrary constant $\rho > 0$. We assume that $\varepsilon_k < \rho/16$ by ignoring terms with small k, and we only consider those CRMs with $|G^k| > \rho - \varepsilon_k$. Since the total spatial variation is uniformly bounded by B, the total number $\tilde{N}^k$ of these CRMs is uniformly bounded in k. Therefore, by using a subsequence of $\{u^k\}$, if necessary, we assume that the number $\tilde{N}^k \equiv \tilde{N} = \tilde{N}(\rho)$ is independent of k. We arrange the $\tilde{N}$ CRMs in order of ascending space coordinates: $\tilde{M}^k_i$, $i = 1, \ldots, \tilde{N}$. By using a subsequence again, if necessary, we assume that

$$\tilde{L}_i \overset{\text{def}}{=} \lim_{k\to\infty} \tilde{L}^k_i \quad \text{and} \quad \tilde{R}_i \overset{\text{def}}{=} \lim_{k\to\infty} \tilde{R}^k_i \tag{7.1}$$

exist. Clearly, $|\tilde{R}_i - \tilde{L}_i| \ge \rho$. By further screening the sequences of the numerical solutions, we construct an ATD from $t = 0$ to $t = 1$ for each sequence $\{\tilde{M}^k_i\}_{k=1}^{\infty}$ of CRMs. Hence, we need a sequence of pairs of ε_k-paths for each of them. Usually, ε_k-E paths play this role. However, we sometimes have to use the following ε_k-paths of the second type for the leftmost or the rightmost ATD. Suppose $P^k(t)$ and $Q^k(t)$ are the two integer-valued functions satisfying

$$x^k_{P^k(t)} < st - 2\varepsilon_k \le x^k_{P^k(t)+1} < x^k_{Q^k(t)-1} \le st + 2\varepsilon_k < x^k_{Q^k(t)}.$$

By using a subsequence, if necessary, we assume that $ph_k < \varepsilon_k$, where p is the size of the stencil of the scheme (see §1). It is easy to check that $x^k_{P^k(t)}$ and $x^k_{Q^k(t)}$ are ε_k-paths of the second type.

Now we are ready to define two ε_k-paths $x^k_{I^k_i(t)}$ and $x^k_{J^k_i(t)}$ for each $\widetilde{M}^k_i$. If

(7.2) $$|u^k_j(1) - w_-| < 2\varepsilon_k \quad \text{for} \quad s - 1 \le x^k_j \le x^k_{p^k_1},$$

we let $x^k_{I^k_1(t)} = x^k_{P^k(t)}$; if

(7.3) $$|u^k_j(1) - w_+| < 2\varepsilon_k \quad \text{for} \quad s + 1 \ge x^k_j \ge x^k_{q^k_{\widetilde{N}}},$$

we let $x^k_{J^k_{\widetilde{N}}(t)} = x^k_{Q^k(t)}$. In all the other cases, $x^k_{p^k_i}$ and $x^k_{q^k_i}$ belong to some E-stencils of u^k at $t = 1$, and using Lemma 6.10 and Lemma 6.14, we can trace them back by extremum paths $E^{k,i,L}_{0,1}$ and $E^{k,i,R}_{0,1}$, respectively, or by their corresponding ε_k-E paths $x^k_{I^k_i(t)}$ and $x^k_{J^k_i(t)}$, respectively.

Using these paths, we define

(7.4) $$\Omega^{(k)}_i \stackrel{\text{def}}{=} \{(x,t) : x^k_{I^k_i(t)-1/2} \le x \le x^k_{J^k_i(t)-1/2},\ \ 0 \le t \le 1\},$$
$$\Omega^{(k)} \stackrel{\text{def}}{=} \textstyle\bigcup_{i=1}^{\widetilde{N}(\rho)} \Omega^{(k)}_i .$$

By Lemma 6.14, for every i between 1 and $\widetilde{N}$, $\{x^k_{I^k_i(t)}, x^k_{J^k_i(t)}\}$ is a pair of ε_k-paths in the sense of Definitions 3.5 and 3.6. It is straightforward to verify that they satisfy the properties (i), (ii) and (iii) in Definition 3.8. Hence, they are ATWs with a common limit path $x = st$. Moreover, if $J^k_{i-1}(t) \le j < j+1 \le I^k_i$, and $|u^k_{j+1}(t) - u^k_j(t)| > \rho + \varepsilon_k$, then (x^k_j, t) and (x^k_{j+1}, t) are between two extremum paths such that their x-projections at $t = 1$ either are identical, or are the two E-stencils of a CRM with a jump bounded by $\rho - \varepsilon_k$ because the CRM does not belong to the collection of $\widetilde{N}$ CRMs. This violates Lemma 6.12, and hence $|u^k_{j+1}(t) - u^k_j(t)| \le \rho + \varepsilon_k$ when $J^k_{i-1}(t) \le j < j+1 \le I^k_i$. This is Property (iii) in Lemma 7.3. Property (ii) can be shown similarly. It remains to show that there is a sequence in $\widetilde{\Psi}_{w_-,w_+,B}$ such that not only the preceding properties hold, but also all the $\widetilde{N}(\rho)$ ATWs are essentially monotone, i.e., they are ATDs. We achieve this goal in the following three steps.

Step 1. We show that for any $t \in [0,1]$, and for sufficiently large k, if the integers p and q satisfy $I^k_i(t) \le p < q \le J^k_i(t)$ where the positive integer $i \le \widetilde{N}(\rho)$, and if $u^k_q(t) - u^k_p(t)$ and $\widetilde{R}_i - \widetilde{L}_i$ have opposite signs, then $|u^k_q(t) - u^k_p(t)| < 6\varepsilon_k$.

By Lemma 6.15, the only nontrivial cases are the ones when at least one of $x^k_{I^k_i(t)}$ and $x^k_{J^k_i(t)}$ is an ε_k-path of the second type. Hence, we focus on the case that $i = 1$, that $x^k_{I^k_1(t)}$ is an ε_k-path of the second type, and that $x^k_{J^k_1(t)}$ is an ε_k-E path. All other cases can be dealt with similarly.

Without loss of generality, assume that $\widetilde{R}_i > \widetilde{L}_i$. We argue by contradiction. Hence, we assume that $u^k_p(t) - u^k_q(t) \ge 6\varepsilon_k$. Then either $u^k_p(t) \ge w_- + 3\varepsilon_k$, or $u^k_p(t) < w_- - 3\varepsilon_k$. Consider the case $u^k_p(t) \ge w_- + 3\varepsilon_k$. Let $E^{k,A}_{0,1}$ and $E^{k,B}_{0,1}$ be two extremum paths with $\max P_{E^{k,A}}(t) < x^k_p$, and $\max P_{E^{k,B}}(t) \ge x^k_p$ such that the two E-stencils $P_{E^{k,A}}(1)$ and $P_{E^{k,B}}(1)$ are either identical or successive. Lemma 6.12 implies that either $E^{k,A}_{0,1}$ is a ridge with $V_{E^{k,A}}(1) > w_- + 2\varepsilon_k$, or $E^{k,B}_{0,1}$ is. The former implies that either $P_{E^{k,A}}(1) = P_{E^{k,1,R}}(1)$, which by Lemma 6.12 leads to $u^k_p(t) - u^k_q(t) < 2\varepsilon_k$, a contradiction to the assumption, or $E^{k,A}_{0,1} \cap \Omega = \emptyset$, which implies that if $(x', t) \in E^{k,A}_{0,1}$ and $(x'', t) \in \Omega^-_{\varepsilon_k}$, then $x' < x'' < x^k_p$, $u^k(x', t) - u^k(x'', t) > \varepsilon_k$, and $u^k_p(t) - u^k(x'', t) > 2\varepsilon_k$. The last three inequalities violate Lemma 6.12. The latter implies that $P_{E^{k,B}}(1) = P_{E^{k,1,R}}(1)$, and that both $E^{k,B}_{0,1}$ and $E^{k,1,R}_{0,1}$ are ridges. Hence, Lemma 6.12 implies that $u^k_p(t) - u^k_q(t) < 2\varepsilon_k$, which leads to a contradiction as mentioned earlier. Similarly, the case $u^k_p(t) < w_- - 3\varepsilon_k$ also leads to contradictions. We let the readers check the details.

Step 2. It is in this step that Assumption 3.3 is needed. First, let us fix an i between 1 and $\widetilde{N}(\rho)$, and consider the sequence of pairs $\{x^k_{P^k(t)}, x^k_{I^k_i(t)}\}$. Clearly, it is an ATW with the limit path $x = st$ if $w_- \ne \widetilde{L}_i$. Since the scheme satisfies Assumption 3.3, and since $s(w_+ - w_-) = f(w_+) - f(w_-)$, an application of Corollary 3.11 to the ATW yields that $\widetilde{L}_i \in \{w : f(w) = f[w; w_-, w_+]\}$ if $w_- \ne \widetilde{L}_i$. However, since $w_- \in \{w : f(w) = f[w; w_-, w_+]\}$, we have that $\widetilde{L}_i \in \{w : f(w) = f[w; w_-, w_+]\}$ always holds. Similarly, $\widetilde{R}_i \in \{w : f(w) = f[w; w_-, w_+]\}$. Consider $\{\widehat{u}^k\} = T^{1/2}_{s/2,1/2}\{u^k\}$. By Lemma 7.1 and the convention made at the beginning of this section, we have

(a) $\{\widehat{u}^k\} \in \widetilde{\Psi}_{w_-,w_+,B}$.

(b) For each $i = 1, \ldots, \widetilde{N}(\rho)$, the sequence of pairs

(7.5) $$\{\widehat{x}_{\widehat{I}^k_i(t)}, \widehat{x}_{\widehat{J}^k_i(t)}\} = \{\widehat{x}_{I^k_i((1+t)/2)}, \widehat{x}_{J^k_i((1+t)/2)}\} = \{2x_{I^k_i((1+t)/2)} - s, 2x_{J^k_i((1+t)/2)} - s\}$$

is an ATW of $\{\widehat{u}^k\}$ with the limit path $x = st$ and the two states $\widetilde{L}_i$ and $\widetilde{R}_i$.

Because the effects of the mapping $T^{1/2}_{1/2,s/2}$ are no more than a scaling and a displacement, the following properties also hold:

(c) If $|\widehat{u}^k(x,t) - W(x,t)| \ge \varepsilon_k$, then $(x,t) \in \Omega^c_{2\varepsilon_k}$.

(d) All the properties (i)–(iii) in Lemma 7.3, with the ATWs replaced by their $S^{1/2}_{s/2,1/2}$ images.

(e) The values of $\{\widehat{u}^k\}$ satisfy the result of the step 1, i.e., $|\widehat{u}^k_q(t) - \widehat{u}^k_p(t)| < 6\varepsilon_k$ when $\widehat{I}^k_i(t) \le p < q \le \widehat{J}^k_i(t)$ and $(\widehat{u}^k_q(t) - \widehat{u}^k_p(t))(\widetilde{R}_i - \widetilde{L}_i) < 0$.

We now prove one more property of $\{\widehat{u}^k\}$:

(f) Let $\{j^k\}$ be a sequence of integers, and $\{\widehat{t}^k\}$ be a sequence of real numbers in $[0,1]$. Suppose that $\widehat{x}^k_{\widehat{I}^k_i(\widehat{t}^k)} \le \widehat{x}^k_{j^k} < \widehat{x}^k_{j^k+1} \le \widehat{x}^k_{\widehat{J}^k_i(\widehat{t}^k)}$ and that $u^k_{j^k+1}(\widehat{t}^k) - u^k_{j^k}(\widehat{t}^k)$ and $\widetilde{R}^k_i - \widetilde{L}^k_i$ have opposite signs. If $\lim_{k\to\infty} u^k_{j^k}(\widehat{t}^k) = A$, then $A \in \{w : f(w) = f[w; w_-, w_+]\}$.

In fact, by letting $t^k = (1 + \widehat{t}^k)$, noticing that $\{u^k|_{t \ge 1/2}\} = (T^{1/2}_{s/2,1/2})^{-1}\{\widehat{u}^k\}$, and using (7.5), we get

$$\{t^k\} \subset [1/2, 1], \quad x^k_{I^k_i(t^k)} \le x^k_{j^k} < x^k_{j^k+1} \le x^k_{J^k_i(t^k)}, \quad \text{and} \quad (u^k_{j^k+1}(t^k) - u^k_{j^k}(t^k))(\widetilde{R}_i - \widetilde{L}_i) < 0.$$

Assume that A does not belong to $\{w : f(w) = f[w; w_-, w_+]\}$. Then by the result of step 1, for sufficiently large k, there is an E-stencil of u^k at t^k which can be traced back by an extremum path

F^k_{0,t^k} or by an ε_k-E path $\bar{z}^k(t)$, such that the following properties hold:

$$\bar{z}^k(t^k) \in P_{E^k}(t^k), \quad |u^k(\bar{z}^k(t^k), t^k) - u^k_{j_*}(t^k)| < 6\varepsilon_k, \quad \text{and} \quad x^k_{I^k(t)} < \bar{z}^k(t) < x^k_{J^k(t)}.$$

Denoting $x^k_{I^k(t)}$ and $x^k_{J^k(t)}$ by $\tilde{x}^k(t)$ and $\tilde{y}^k(t)$, respectively, and applying Lemma 7.1, one sees that if $x^k(t) = 2\tilde{x}^k(t/2)$ and $z^k(t) = 2\bar{z}^k(t/2)$, then the pair $\{x^k(t), z^k(t)\}$ is an ATW of $T^{1/2}_{0,0}\{u^k\} \in \Psi_{w_-,w_+,B}$ with the limit path $x = st$ and the two states $\tilde{L}_i$ and A. Hence, by Corollary 3.11, $A \in \{w : f(w) = f[w; w_-, w_+]\}$. This contradiction proves the property (f).

Step 3. Now we construct the desired sequence, still denoted by $\{u^k\}$, in $\Psi_{w_-,w_+,B}$ as follows.

By (f), given any $\eta > 0$, there is a $K(\eta)$ such that if $t \in [0,1]$, and $i = 1, \ldots, \tilde{N}(\rho)$, and if j is an integer such that $I^k_i(t) \le j < j+1 \le J^k_i(t)$ and that $(\hat{u}^k_{j+1}(t) - \hat{u}^k_j(t))(\tilde{R}_i - \tilde{L}_i) < 0$, then

$$\hat{u}^k_j(t), \hat{u}^k_{j+1}(t) \in \mathcal{N}_\eta(\{w : f(w) = f[w; w_-, w_+]\})$$

provided $k > K(\eta)$. Let $\{\varepsilon_{m_k}\}$ be a subsequence of $\{\varepsilon_k\}$ such that $\varepsilon_m < 6\varepsilon_k$ when $m > m_k$. Let $n_k = \max(K_{\varepsilon_k}, m_k)$. Finally, define $u^k = \hat{u}^{n_k}$. From the properties (a) to (f), it is evident that the sequence $\{u^k\}$ has all the properties stated in Lemma 7.3. □

8. PROOF OF THEOREM 3.13

Again we argue by contradiction. Assume that a self-similar scheme of the form (1.2)-(1.4) that satisfies Assumption 3.3 fails to converge. By Theorem 3.2 there are two distinct numbers w_- and w_+ and a positive constant B such that the set $\hat{\Psi}_{w_-,w_+,B}$ is not empty, and such that its limit W is a traveling expansion shock. Therefore, by definition, there exists a convex entropy function $U(w)$ with its corresponding flux $F(w)$ such that

$$\Phi = \int_{w_-}^{w_+} U''(w)(f[w; w_-, w_+] - f(w))dw \int_{x=st} \phi(x,t)dt > 0,$$

where $\phi(x,t)$ is a nonnegative smooth test function with compact support in the interior of Ω such that $\int_{x=st}\phi(x,t)dt > 0$. Hence the set $\Psi_{w_-,w_+,B}$ is also not empty and contains a sequence $\{u^k(x,t)\}$ that satisfies Lemma 7.3. We keep all the notation from Lemma 7.3 and assume that $\{\Omega^{(k)}_i\}^\infty_{k=1}$, $i = 1, \ldots, \tilde{N}(\rho)$ and $\Omega^{(k)}$ are defined by (7.4). Our goal is to show that at least one of the ATDs is an ATES. We consider

$$\Phi^k = \int_0^1 \sum_j h_k\Big(\frac{d}{dt}U(u^k_j(t)) + D_+G^k_{j-\frac{1}{2}}(t)\Big)\varphi_{j+\frac{1}{2}}(t)dt, \tag{8.1}$$

where $\varphi_{j+\frac{1}{2}}(t) = \phi(x^k_{j+\frac{1}{2}}, t)$. Integrating by parts and passing to the limit, one shows easily that Φ^k approaches Φ. We split Φ^k into two parts:

$$\Phi^k = \Phi^k_1 + \Phi^k_2,$$

where Φ^k_2 denotes the summation over $\Omega^{(k)}$, and Φ^k_1 denotes the summation that is not included in Φ^k_2. We have

$$|\Phi^k_1| \le U''_{\max}\phi_{\max}\int_0^1 \sum_{(x^k_j(t),t)\in\Omega\setminus\Omega^{(k)}} \left|\int_{u^k_j(t)}^{u^k_{j+1}(t)} (g^k_{j+\frac{1}{2}}(t) - f(w))dw\right| dt.$$

By Corollary 3.4, if either

$$|\Delta_+u^k_j(t)| + |\Delta_+u^k_{j+1}(t)| < 2(\rho + 2\varepsilon_k) \quad \text{or} \quad |\Delta_+u^k_{j-1}(t)| + |\Delta_+u^k_j(t)| < 2(\rho + \varepsilon_k). \tag{8.2}$$

then $|g^k_{j+\frac{1}{2}}(t) - f(w)| < C\rho$ when w is between $u^k_j(t)$ and $u^k_{j+1}(t)$. If neither inequality in (8.2) holds, then there is an i between 1 and $\tilde{N}(\rho) - 1$ such that $j = J^k_i(t) = I^k_{i+1}(t) - 1$. Hence,

$$\begin{aligned}
|\Phi^k_1| \;\le\;& U''_{\max}\phi_{\max}\int_0^1 C\rho\sum_{j=-\infty}^{\infty}|\Delta_+u^k_j(t)| \\
&+ U''_{\max}\phi_{\max}\int_0^1\sum_{i=1}^{\tilde{N}(\rho)-1}\left|\int_{u^k_{J^k_i(t)}(t)}^{u^k_{J^k_i(t)+1}(t)}(g^k_{J^k_i(t)+1/2}(t) - f(u^k_{J^k_i(t)}(t)))dw\right|dt \\
&+ U''_{\max}\phi_{\max}\int_0^1\sum_{i=1}^{\tilde{N}(\rho)-1}\left|\int_{u^k_{J^k_i(t)}(t)}^{u^k_{J^k_i(t)+1}(t)}(f(u^k_{J^k_i(t)}(t)) - f(w))dw\right|dt.
\end{aligned}$$

The first and the third terms on the right are bounded by $CU''_{\max}\phi_{\max}B\rho$. By Lemma 3.7, the second term is bounded by

$$U''_{\max}\phi_{\max}(\rho + \varepsilon_k)\sum_{i=1}^{\tilde{N}(\rho)-1}\int_0^1\left|g^k_{J^k_i(t)+1/2}(t) - f(u^k_{J^k_i(t)}(t))\right|dt \le CU''_{\max}\phi_{\max}B\varepsilon_k \to 0$$

as $k \to \infty$, because $\tilde{N}(\rho)(\rho + \varepsilon_k) \le CB$. Hence, there is a constant $\tilde{C}$ independent of ρ and k such that $|\Phi^k_1| < \tilde{C}\rho$ for sufficiently large k.

On the other hand, Lemma 3.10 implies that

$$\Phi^k_2 \to \Phi_2 = \sum_{i=1}^{\tilde{N}}[F(\tilde{R}_i) - F(\tilde{L}_i) - s(U(\tilde{R}_i) - U(\tilde{L}_i))]\int_{x=st}\phi(x,t)dt.$$

Hence, we have

$$\Phi_2 \ge \Phi - \tilde{C}\rho. \tag{8.3}$$

It follows for sufficiently small ρ that $\Phi_2 > 0$ and hence there exists a positive integer $i \le \tilde{N}(\rho)$ such that

$$F(\tilde{R}_i) - F(\tilde{L}_i) - s(U(\tilde{R}_i) - U(\tilde{L}_i)) > 0. \tag{8.4}$$

Therefore, $\{u^k\}$ harbors an ATES. □

Appendix. The purpose of this appendix is to prove Lemma 7.2. Consider a semidiscrete scheme that satisfies a local maximum principle in the sense of Assumption 6.4. We begin with the following

Lemma A.1 ($L^\infty(L^1_{loc})$ Convergence). *For any sequence $\{u_k\} \in \widehat{\Psi}_{w_-,w_+,B}$ with the limit $W(x,t)$, and for any fixed $\alpha > 0$,*

$$\int_{st-\alpha}^{st+\alpha} |u^k(x,t) - W(x,t)|dx \to 0 \quad as \quad k \to \infty \tag{A.1}$$

uniformly for $t \in [t_a, t_b]$. Here, $W(x,t)$ is the limit of $\widehat{\Psi}_{w_-,w_+,B}$, and $[t_a,t_b]$ is a fixed interval.

Proof. In this proof, C stands for a generic constant. We assume the contrary, namely, that there exists a positive constant η, a sequence of integers $\{k_j\}_{j=1}^\infty$ with $k_{j+1} > k_j$, and a sequence $\{t_j\}_{j=1}^\infty$ in $[t_a,t_b]$ such that

$$\int_{st_j-\alpha}^{st_j+\alpha} |u^{k_j}(x,t_j) - W(x,t_j)|dx > \eta. \tag{A.2}$$

However, since $TV_{u^k}(t) < B$, and since the numerical flux is Lipschitz continuous,

$$\begin{aligned} \| u^{k_j}(t_1) - u^{k_j}(t_2) \|_{L^1} &= \sum_{i=-\infty}^{\infty} |u_i^{k_j}(t_2) - u_i^{k_j}(t_1)|h_{k_j} \\ &= \sum_{i=-\infty}^{\infty} \left| \int_{t_1}^{t_2} h_{k_j} \frac{du_i^{k_j}(t)}{dt} dt \right| \\ &= \sum_{i=-\infty}^{\infty} \left| \int_{t_1}^{t_2} (g_{i+1/2}^{k_j}(t) - g_{i-1/2}^{k_j}(t))dt \right| \\ &\le C|t_1 - t_2| \end{aligned}$$

for any t_1 and t_2. Because W has the same property, the triangle inequality implies that $\delta u^{k_j} \stackrel{\text{def}}{=} u^{k_j} - W$ satisfies

$$\| \delta u^{k_j}(t_1) - \delta u^{k_j}(t_2) \|_{L_1} \le C|t_1 - t_2|.$$

Therefore,

$$\begin{aligned} &\left| \textstyle\int_{st_2-\alpha}^{st_2+\alpha} |\delta u^{k_j}(t_2)|dx - \int_{st_1-\alpha}^{st_1+\alpha} |\delta u^{k_j}(t_1)|dx \right| \\ &\le \| \delta u^{k_j}(t_1) - \delta u^{k_j}(t_2) \|_{L_1} + C|t_1 - t_2| \\ &\le C|t_2 - t_1|. \end{aligned} \tag{A.3}$$

The relations (A.2) and (A.3) imply that

$$\int_{t_a-\eta/C}^{t_b+\eta/C} dt \int_{st-\alpha}^{st+\alpha} |\delta u^{k_j}(x,t)|dx \ge \frac{\eta^2}{C}.$$

This contradicts the L^1_{loc} convergence of u^k. □

Assume that the set $\widehat{\Psi}_{w_-,w_+,B}$ is not empty. Our proof of Lemma 7.2 consists of 2 steps.

Step 1. We first construct a finite collection of paths that attract the set

$$\{(x,t) \in \Omega_\delta^- : u^k(x,t)- > w_- + \rho\}.$$

For any k with $\varepsilon_k < \rho/16$, set $\alpha_k = w_- + \rho/2 + \varepsilon_k$ and $\beta_k = w_- + \rho - \varepsilon_k$. We divide the set of all subscripts j with $u_j^k(1) > \beta_k$ into disjoint subsets:

$$\{j : \quad u_j^k(1) > \beta_k\} = \bigcup_{m=1}^{\widetilde{m}_k} \Pi_m^k$$

such that:

(a) If $\widetilde{m}_k \ge 2$, then $\max \Pi_m^k < \min \Pi_{m+1}^k$ when $m = 1,2,\ldots,\widetilde{m}_k - 1$.

(b) If $\widetilde{m}_k \ge 2$, then for each $m = 1,2,\ldots,\widetilde{m}_k - 1$, there is a integer j between Π_m^k and Π_{m+1}^k for which $u_j^k(1) < \alpha_k$.

(c) For each $m = 1,2,\ldots,\widetilde{m}_k$, $u_j^k(1) \ge \alpha_k$ if $\min \Pi_m^k \le j \le \max \Pi_m^k$.

We call the set $x_{\Pi_m^k}$ *a ρ-peak* of u^k. The following properties follow immediately:

1. Since $TV_{u^k}(t) < B$ (see §3), the numbers $\widetilde{m}_k$ are bounded independent of k:

$$\widetilde{m}_k < \frac{B}{\rho - 4\varepsilon_k} + 1 < \frac{4B}{3\rho} + 1.$$

2. For each m between 1 and $\widetilde{m}_k$, fix one ridge $M_{0,1}^{k,m}$, if any, such that its x-projection $P_{M^{k,m}}(1)$ at $t = 1$ is a subset of $x_{\Pi_m^k}$ and call it *the marked ridge* m. It may happen that the marked ridge 1 or marked ridge $\widetilde{m}_k$ does not exist. In such a case, Π_1^k or $\Pi_{\widetilde{m}_k}^k$ must be an infinite set of consecutive integers, and $u_j^k(1)$ must be monotone in j for $j \in \Pi_1^k$ or for $j \in \Pi_{\widetilde{m}_k}^k$, i.e., $u_j^k(1)$ is greater than β_k and is increasing in j for sufficiently large j, or $u_j^k(1)$ is greater than β_k and is decreasing for sufficiently small j.

The following lemma confirms that the union of these marked ridges attract the set $\{(x,t) \in \Omega_\delta^- : u^k(x,t) > w_- + \rho\}$.

Lemma A.2. *For any constants δ and ε with $0 < \varepsilon < \delta/2$, there exists a $K > 0$ such that for $k > K$, if $(y,t_0) \in \Omega_\delta^-$ and*

$$u^k(y,t_0) > w_- + \rho, \tag{A.4}$$

then there exists an m between 1 and $\widetilde{m}_k$ such that the marked ridge m exists, and such that $|y - x_j^k| < \varepsilon$ for any $x_j^k \in P_{M^{k,m}}(t_0)$.

Proof. According to the $L^\infty(L^1_{loc})$ Convergence Lemma A.1 (with $\alpha = 1$), for $\varepsilon \in (0,\delta/2)$ there exists a real number K such that for $k > K$, there exist x' and x'' with $y - \varepsilon < x' < y < x'' < y + \varepsilon$ such that

$$u^k(x',t_0) < w_- + \rho/2 \quad \text{and} \quad u^k(x'',t_0) < w_- + \rho/2. \tag{A.5}$$

Assume that $E_{0,1}^{k,L}$ and $E_{0,1}^{k,R}$ are two extremum paths of u^k between $t = 0$ and $t = 1$ such that $\max P_{E^{k,L}}(t_0) < y \le \max P_{E^{k,R}}(t_0)$ and such that $\max P_{E^{k,L}}(1)$ and $P_{E^{k,R}}(1)$ are either identical or successive E-stencils of u^k at $t = 1$. Without loss of generality, assume that $E_{0,1}^{k,L} = N_{0,1}^{k,L}$ is a trough and $E_{0,1}^{k,R} = M_{0,1}^{k,R}$ is a ridge. Noticing (A.4) and the second inequality of (A.5), and applying Lemma 6.12, we see that $\max P_{M^{k,R}}(t_0) < x''$ and that $V_{M^{k,R}}(1) > \beta_k$. Therefore, $P_{M^{k,R}}(1) \subset x^k_{\Pi^{k,m}}$ for some integer m between 1 and $\tilde{m}_k$. Hence, the marked ridge m, $M_{0,1}^{k,m}$, exists. The marked ridge m and $M_{0,1}^{k,R}$ may or may not be the same ridge. Nevertheless, since both $P_{M^{k,R}}(1)$ and $P_{M^{k,m}}(1)$ belong to $x^k_{\Pi^{k,m}}$, Lemma 6.12 and the inequalities in (A.5) yield the desired relation $P_{M^{k,m}}(t_0) \in (x', x'')$.

In the case there are not two extremum paths $E_{0,1}^{k,L}$ and $E_{0,1}^{k,R}$ with the aforementioned properties, the conclusion of the lemma still holds (see the Remark at the end of §6). □

Step 2. By finding a special sequence $\{u^k\}$ in $\hat{\Psi}_{w_-,w_+,B}$, we will separate and concentrate the sequence of the sets $\{\{(x,t) \in \Omega : u^k(x,t) > w_- + \rho,\ x \le st\}\}_{k=1}^{\infty}$. We use the same notation $\{u^k\}$ for all the sequences involved unless otherwise specified. Since for fixed ρ, the sequence $\{\tilde{m}_k\}$ is uniformly bounded, we assume, by using a subsequence, if necessary, that $\tilde{m}_k \equiv \tilde{m}$ for all k. With this sequence and for each k and $m = 1, \ldots, \tilde{m}$, define successively

$$z^{k,m}(t) \stackrel{\text{def}}{=} \min P_{M^{k,m}}(t) - st,$$
$$Z^{k,m} \stackrel{\text{def}}{=} \inf_{t \in [0,1]} z^{k,m}(t),$$
$$Z^m \stackrel{\text{def}}{=} \lim_{k\to\infty} \inf Z^{k,m}.$$

Obviously, Z^m increases with m. By using a subsequence again, if necessary, we assume that

$$Z^m = \lim_{k\to\infty} Z^{k,m}.$$

Let $\overline{m}$ be the integer, if any, such that $1 \le \overline{m} \le \tilde{m}$ for which $Z^{\overline{m}} < 0$ and $Z^{\overline{m}+1} \ge 0$.

Without loss of generality, assume that $1 \le \overline{m} < \tilde{m}$. By using a subsequence, if necessary, we assume that there is a sequence $\{t^k\} \subset [0,1]$, with $t^k \to \bar{t}$, such that $z^{k,\overline{m}}(t^k) \to Z^{\overline{m}}$ as k tends to infinity.

Lemma A.3. *There exists a constant $\gamma > 0$, independent of k, such that for sufficiently large k, the part $M^{k,\overline{m}}_{\bar{t}-\gamma,\bar{t}+\gamma}$ of the marked ridge m which is of course not affected by using subsequences, is entirely to the left of the line $x = st + \frac{3}{4}Z^{\overline{m}}$.*

Proof. The idea is as follows. When k becomes large, $P_{M^{k,\overline{m}}}(t^k)$ keeps a distance to the left of the line $x = st + \frac{3}{4}Z^{\overline{m}}$. Since s is finite, and since the ridge sweeps over all the intermediate points as it moves (Lemma 6.9), it takes some time for them to meet. Until then, the path stays to the left of the line.

Let $\sigma = |Z^{\overline{m}}|/16$, and $\gamma_0 = 2\sigma/|s|$. We have

$$x^R \stackrel{\text{def}}{=} s\bar{t} + Z^{\overline{m}} + 2\sigma < st + \frac{3}{4}Z^{\overline{m}},$$

if $|t - \bar{t}| < \gamma_0$. However, by Lemma A.1,

$$\lim_{k\to\infty} \max P_{M^{k,\overline{m}}}(t^k) = s\bar{t} + Z^{\overline{m}},$$

which implies that, for sufficiently large k,

$$x^L \stackrel{\text{def}}{=} s\bar{t} + Z^{\overline{m}} + \sigma > \max P_{M^{k,\overline{m}}}(t^k). \tag{A.6}$$

Hence, by the Sweeping Over Lemma 6.9, if at a certain time $\hat{t}$, $\max P_{M^{k,\overline{m}}}(\hat{t}) \ge x^R$, then for any $x \in [x^L, x^R]$, there is a time $t(x)$ between t^k and $\hat{t}$ such that

$$u^k(x, t(x)) = V_{M^{k,\overline{m}}}(t(x)) > w_- + \rho - \varepsilon_k \ge w_- + \frac{15}{16}\rho.$$

Therefore,

$$\int_{t^k}^{\hat{t}} \left|\frac{\partial u^k(x,t)}{\partial t}\right| dt \ge (w_- + \frac{15}{16}\rho - u^k(x,\hat{t}))_+, \tag{A.7}$$

where $(a)_+ = \frac{a+|a|}{2}$, and without loss of generality, we have assumed that $\hat{t} > t^k$. On the one hand,

$$\begin{aligned} S_k &\stackrel{\text{def}}{=} \int_{-\infty}^{\infty} dx \int_{t^k}^{\hat{t}} \left|\frac{\partial u^k(x,t)}{\partial t}\right| dt \\ &= \int_{t^k}^{\hat{t}} \Big(\sum_{j=\infty}^{\infty} |g^k_{j+\frac{1}{2}}(t) - g^k_{j-\frac{1}{2}}(t)|\Big) dt \\ &\le cB|t^k - \hat{t}| \end{aligned} \tag{A.8}$$

for some constant c. Now, (A.7) and Lemma A.1 imply that

$$\begin{aligned} S_k &> \textstyle\int_{x^L}^{x^R} (w_- + \frac{15}{16}\rho - u^k(x,\hat{t}))_+ dx \\ &\to \tfrac{15}{16}\rho(x^R - x^L) = \tfrac{15}{16}\sigma\rho \end{aligned}$$

as $k \to \infty$, and hence, for sufficiently large k,

$$S_k > \frac{\sigma\rho}{2}. \tag{A.9}$$

Combining (A.8) and (A.9), we get

$$|\hat{t} - t^k| > \frac{\sigma\rho}{2cB}.$$

Now if we choose $\gamma = \min(\frac{\sigma\rho}{4cB}, \frac{\gamma_0}{2})$, then there exists a real number K such that, when $k > K$, (A.6), (A.9) and $|t^k - \bar{t}| < \gamma$ hold. We then have

$$|t^k - t| < 2\gamma = \min(\frac{\sigma\rho}{2cB}, \gamma_0),$$

and hence, $\max P_{M^{k,\overline{m}}}(t) < x^R < st + 3Z_{\overline{m}}/4$ when $k > K$ and $|t - \bar{t}| < \gamma$. □

According to Lemmas A.2 and A.3, for any $\delta' \in (0, |Z^{\overline{m}}|/4)$ and $\varepsilon \in (0, |Z^{\overline{m}}|/4))$, there exists a $\widetilde{K}$ such that if

$$st + Z^{\overline{m}}/2 \le x \le st - \delta' \quad \text{and} \quad |t - \bar{t}| < \gamma, \quad \text{then}$$

$$u^k(x,t) < w_- + \rho \quad \text{for} \quad k > \widetilde{K}. \tag{A.10}$$

Let $\tau = \min(|Z^{\overline{m}}|/2, \gamma, 1)$. For each k, define $\tilde{u}^k$ by

$$\tilde{u}^k \stackrel{\text{def}}{=} T^{\tau}_{s(\bar{t}-\tau),\bar{t}-\tau} u^k \quad \text{if} \quad \bar{t} - \tau \geq 0$$

and

$$\tilde{u}^k = T^{\tau}_{s\bar{t},\bar{t}} u^k \quad \text{if} \quad \bar{t} - \tau < 0.$$

Then, Lemma 7.1 implies that $\{\tilde{u}^k\} \in \widehat{\Psi}_{w_-,w_+,B}$. By (A.10), $\tilde{u}^k(x,t) < w_- + \rho$ when $k > \widetilde{K}$ and $(x,t) \in \Omega_\delta^-$, where $\delta = \delta'/\tau$. Moreover, since δ' is arbitrary, δ is also arbitrary. Denoting the new sequence by $\{u^k\}$ again, we therefore have shown that for any positive constants δ and ρ, there is a sequence $\{u^k\} \in \widehat{\Psi}_{w_-,w_+,B}$, and there exists $K_{\delta,\rho} > 0$, such that when $k > K_{\delta,\rho}$ and $(x,t) \in \Omega_\delta^-$, one has $u_k(x,t) \leq w_- + \rho$. Ignoring the terms with $k \leq K_{\delta,\rho}$, we have shown that for any positive constants δ and ρ, there is a sequence $\{u^k\} \in \widehat{\Psi}_{w_-,w_+,B}$ such that $u_k(x,t) \leq w_- + \rho$ when $(x,t) \in \Omega_\delta^-$. We can enhance the above result by similarly separating or concentrating the sets $\{(x,t) \in \Omega_\delta^- : u^k(x,t) < w_- - \rho\}$, $\{(x,t) \in \Omega_\delta^+ : u^k(x,t) > w_+ + \rho\}$ and $\{(x,t) \in \Omega_\delta^+ : u^k(x,t) < w_+ - \rho\}$. Summarizing these results, we conclude that, for any pair (δ,ρ) of positive constants, there is a sequence $\{u^k_{\delta,\rho}\} \in \widehat{\Psi}_{w_-,w_+,B}$ such that, if $|u^k_{\delta,\rho}(x,t) - W(x,t)| > \rho$, then $(x,t) \in \Omega_\delta^c$. Since both δ and ρ are arbitrary, Lemma 7.2 follows from a diagonal process.